CAMBRIDGE ASSESSMENT

STEP Solutions
2009

Mathematics
STEP 9465, 9470, 9475

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STEP II, Solutions
2009

1 Both graphs are symmetric in the lines $y = \pm x$, and $x^4 + y^4 = u$ is also symmetric in the x- and y-axes. These facts immediately enable us to write down the coordinates of $B(\beta, \alpha)$, $C(-\alpha, -\beta)$ and $D(-\beta, -\alpha)$. Remember to keep the cyclic order A, B, C, D correct, else this could lead to silly calculational errors later on. The easiest way to show that $ABCD$ is a rectangle is to work out the gradients of the four sides (which turn out to be either 1 or –1) and then note that each pair of adjacent sides is perpendicular using the "product of gradients = –1" result. Working with distances is also a possible solution-approach but, on its own, only establishes that the quadrilateral is a parallelogram. However, the next part requires you to calculate distances anyhow, and we find that CB, DA have length $(\alpha + \beta)\sqrt{2}$ while BA, DC are of length $(\alpha - \beta)\sqrt{2}$. Multiplying these then give the area of $ABCD$ as $2(\alpha^2 - \beta^2)$.

All of this is very straightforward, and the only tricky bit of work comes next. It is important to think of α and β as particular values of x and y satisfying each of the two original equations. It is then clear that $(\alpha^2 - \beta^2)^2 = \alpha^4 + \beta^4 - 2(\alpha^2\beta^2) = u - 2v^2$, so that Area $ABCD = 2\sqrt{u - 2v^2}$. Substituting $u = 81$, $v = 4$ into this formula then gives Area $= 2\sqrt{81 - 2\times16} = 14$, which is intended principally as a means of checking that your answer is correct.

2(i) It is perfectly possible to differentiate a^(sin[πe^x]) by using the *Chain Rule* (on a function of a function of a function) but simplest to take logs. and use implicit differentiation. Then, setting $\frac{dy}{dx} = 0$ and noting that πe^x and $\ln a$ are non-zero, we are left solving the eqn. $\cos(\pi e^x) = 0$ for the turning points. This gives $e^x = (2n+1)\frac{1}{2}\pi \Rightarrow x = \ln\left(n + \frac{1}{2}\right)$, $y = a$ or $\frac{1}{a}$, depending upon whether n is even or odd. Although not actually required at this point, it may be helpful to note at this stage that the evens give maxima while the odds give minima. There is, however, a much more obvious approach to finding the TPs that doesn't require differentiation at all, and that is to use what should be well-known properties of the sine function: namely, that $y = a^{\sin(\pi.\exp x)}$ has maxima when $\sin(\pi e^x) = 1$, i.e. $\pi e^x = (2n + \frac{1}{2})\pi$, and $x = \ln(2n + \frac{1}{2})$ for $n = 0, 1, \dots$, with $y_{max} = a$. Similarly, minima occur when $\sin(\pi e^x) = -1$, i.e. $\pi e^x = (2n - \frac{1}{2})\pi$, and $x = \ln(2n - \frac{1}{2})$ for $n = 1, 2, \dots$, with $y_{min} = \frac{1}{a}$.

(ii) Using the addition formula for $\sin(A + B)$, and the approximations given, we have

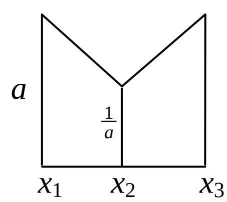
$$\sin(\pi e^x) \approx \sin(\pi + \pi x) = -\sin(\pi x) \approx -\pi x \text{ for small } x,$$

leading to $y \approx a^{-\pi x} = e^{-\pi x \ln a} \approx 1 - \pi x. \ln a$.

(iii) Firstly, we can note that, for $x < 0$, the curve has an asymptote $y = 1$ (as $x \to -\infty$, $y \to 1+$). Next, for $x > 0$, the curve oscillates between a and $\frac{1}{a}$, with the peaks and troughs getting ever closer together. The work in (i) helps us identify the TPs: the first max. occurs when $n = 0$ at a *negative* value of x [N.B. $\ln\left(\frac{1}{2}\right) < 0$] at $y = a$; while the result in (ii) tells us that the curve is approximately negative linear as it crosses the y-axis.

(iv) The final part provides the only really tricky part to the question, and a quick diagram might be immensely useful here. Noting the relevant x-coordinates

$$x_1 = \ln\left(2k - \tfrac{3}{2}\right),\ x_2 = \ln\left(2k - \tfrac{1}{2}\right), \text{ and } x_3 = \ln\left(2k + \tfrac{1}{2}\right),$$



the area is the sum of two trapezia (or rectangle – triangle), and manipulating $\ln\left(\frac{4k+1}{4k-3}\right) = \ln\left(\frac{4k-3+4}{4k-3}\right) = \ln\left(1 + \frac{1}{k - \frac{3}{4}}\right)$ leads to the final, given answer.

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3 Using the "addition" formula for $\tan(A - B)$,

$$\text{LHS} \equiv \tan\left(\tfrac{\pi}{4} - \tfrac{x}{2}\right) \equiv \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}} \equiv \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} \equiv \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} \times \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}$$

$$\equiv \frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \text{ (since } c^2 + s^2 = 1) \equiv \frac{1 - \sin x}{\cos x} \equiv \sec x - \tan x \equiv \text{RHS}.$$

Alternatively, one could use the "$t = \tan(\tfrac{1}{2}\text{-angle})$" formulae to show that

$$\text{RHS} \equiv \frac{1 - t^2}{1 + t^2} - \frac{2t}{1 - t^2} \equiv \frac{(1 - t)^2}{(1 - t)(1 + t)} \equiv \frac{1 - t}{1 + t} \equiv \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}} \equiv \text{LHS}.$$

(i) Setting $x = \frac{\pi}{4}$ in (*) $\Rightarrow \tan\frac{\pi}{8} = \sqrt{2} - 1$. Then, using the addition formula for $\tan(A + B)$ with $A = \frac{\pi}{3}$ and $B = \frac{\pi}{8}$, we have $\tan\frac{11\pi}{24} = \tan\left(\frac{\pi}{3} + \frac{\pi}{8}\right) = \dfrac{\sqrt{3} + \sqrt{2} - 1}{1 - \sqrt{3}\left(\sqrt{2} - 1\right)} = \dfrac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{3} - \sqrt{6} + 1}$, as required.

(ii) Now, in the "spirit" of maths, one might reasonably expect that one should take the given expression, rationalise the denominator (twice) and derive the given answer, along the lines …

$$\frac{\sqrt{3} + \sqrt{2} - 1}{1 + \sqrt{3} - \sqrt{6}} = \frac{\sqrt{3} + \sqrt{2} - 1}{1 + \sqrt{3} - \sqrt{6}} \times \frac{1 + \sqrt{3} + \sqrt{6}}{1 + \sqrt{3} + \sqrt{6}} = \frac{1 + 2\sqrt{2} + \sqrt{3}}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = 2 + \sqrt{2} + \sqrt{3} + \sqrt{6}.$$

However, with a given answer, it is perfectly legitimate merely to multiply across and verify that $\left(\sqrt{3} - \sqrt{6} + 1\right)\left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right) = \sqrt{3} + \sqrt{2} - 1$.

(iii) Having got this far, the end is really very clearly signposted. Setting $x = \frac{11\pi}{24}$ in (*) gives

$$\tan\tfrac{\pi}{48} = \sec\tfrac{11\pi}{24} - \tan\tfrac{11\pi}{24} = \sqrt{1 + t^2} - t$$

$$= \sqrt{1 + \left[4 + 2 + 3 + 6 + 4\sqrt{2} + 4\sqrt{3} + 4\sqrt{6} + 2\sqrt{6} + 2\sqrt{12} + 2\sqrt{18}\right]} - \left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right)$$

$$= \sqrt{15 + 10\sqrt{2} + 8\sqrt{3} + 6\sqrt{6}} - \left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right)$$

4(i) Writing $p(x) - 1 \equiv q(x).(x - 1)^5$, where $q(x)$ is a quartic polynomial, immediately gives $p(1) = 1$.

(ii) Diff^g. using the product and chain rules leads to

$$p'(x) \equiv q(x).5(x - 1)^4 + q'(x).(x - 1)^5 \equiv (x - 1)^4.\{5\,q(x) + (x - 1)\,q'(x)\},$$

so that $p'(x)$ is divisible by $(x - 1)^4$.

(iii) Similarly, we have that $p'(x)$ is divisible by $(x + 1)^4$ and $p(-1) = -1$.
Thus $p'(x)$ is divisible by $(x + 1)^4.(x - 1)^4 \equiv (x^2 - 1)^4$. However, $p'(x)$ *is* a polynomial of degree eight, hence $p'(x) \equiv k(x^2 - 1)^4$ for some constant k . That is, $p'(x) \equiv k\left(x^8 - 4x^6 + 6x^4 - 4x^2 + 1\right)$.
Integrating term by term then gives $p(x) \equiv k\left(\frac{1}{9}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\right) + C$, and use of both $p(1) = 1$ and $p(-1) = -1$ help to find k and C; namely, $k = \dfrac{315}{128}$ and $C = 0$.

5 The very first bit is not just a giveaway mark, but rather a helpful indicator of the kind of result or technique that may be used in this question: $\left(\sqrt{x-1}+1\right)^2 = x + 2\sqrt{x-1}$; but pay attention to what happens here. Most particularly, the fact that $\left(\sqrt{x-1}+1\right)^2 = x + 2\sqrt{x-1}$ does NOT necessarily mean that $\sqrt{x+2\sqrt{x-1}} = \sqrt{x-1}+1$ since positive numbers have *two* square-roots! Recall that $\sqrt{x^2} = |x|$ and not just *x*. Notice that, during the course of this question, the range of values under consideration switches from (5, 10) to $\left(\frac{5}{4}, 10\right)$, and one doesn't need to be particularly suspicious to wonder why this is so. A modicum of investigation at the outset seems warranted here, as to when things change sign.

(i) So … while $\sqrt{x+2\sqrt{x-1}} = \sqrt{x-1}+1$ seems a perfectly acceptable thing to write, since $x \ge 1$ is a necessary condition in order to be able to take square-roots at all here (for real numbers), simply writing down that $\sqrt{x-2\sqrt{x-1}} = +\left(\sqrt{x-1}-1\right)$ *may* cause a problem. A tiny amount of exploration shows that $\sqrt{x-1}-1$ changes from negative to positive around $x = 2$. Hence, in part (i), we can ignore any negative considerations and plough ahead: $I = \int_5^{10} 2\,dx = \left[2x\right]_5^{10} = 10$.

(ii) Here in (ii), however, you should realise that the area requested is the sum of two portions, one of which lies below the *x*-axis, and would thus contribute negatively to the total if you failed to take this into account. Thus,

$$\text{Area} = \int_{1.25}^{2} \frac{1-\sqrt{x-1}}{\sqrt{x-1}}\,dx + \int_{2}^{10} \frac{\sqrt{x-1}-1}{\sqrt{x-1}}\,dx = \int_{1.25}^{2}\left[(x-1)^{-\frac{1}{2}}-1\right]dx + \int_{2}^{10}\left[1-(x-1)^{-\frac{1}{2}}\right]dx$$

$$= \left[2\sqrt{x-1}-x\right]_{1.25}^{2} + \left[x-2\sqrt{x-1}\right]_{2}^{10} = 4\tfrac{1}{4}\ .$$

(iii) Now $\left(\sqrt{x+1}-1\right)^2 = x+2-2\sqrt{x+1}\ \ \forall\, x \ge 0$ so we have no cause for concern here. Then

$$I = \int_{x=1.25}^{10} \frac{1+\sqrt{x-1}+\sqrt{x+1}-1}{\sqrt{x-1}\sqrt{x+1}}\,dx = \int_{x=1.25}^{10}\left((x+1)^{-\frac{1}{2}}+(x-1)^{-\frac{1}{2}}\right)dx$$

$$= \left[2\sqrt{x+1}+2\sqrt{x-1}\right]_{1.25}^{10} = 2\left(\sqrt{11}+1\right)$$

6 If you don't know about the *Fibonacci Numbers* by now, then … shame on you! Nevertheless, the first couple of marks for writing down the next few terms must count as among the easiest on the paper. ($F_1 = 1$, $F_2 = 1$), $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$ and $F_{10} = 55$.

(i) If you're careful, the next section isn't particularly difficult either. Using the recurrence relation gives $\frac{1}{F_i} = \frac{1}{F_{i-1}+F_{i-2}} > \frac{1}{2F_{i-1}}$ since $F_{i-2} < F_{i-1}$ for $i \ge 4$. Splitting off the first few terms then leads to $S = \sum_{i=1}^{n}\frac{1}{F_i} > \frac{1}{F_1}+\frac{1}{F_2}\left(1+\frac{1}{2}+\frac{1}{4}+\ldots..\right)$ or $\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_3}\left(1+\frac{1}{2}+\frac{1}{4}+\ldots..\right)$, where the long bracket at the end is the sum-to-infinity of a GP. These give, respectively, $S > 1 + 1 \times 2 = 3$ or $1 + 1 + \frac{1}{2}\times 2 = 3$. A simpler approach could involve nothing more complicated than adding the terms until a sum greater than 3 is reached, which happens when you reach F_5.

A similar approach yields $\frac{1}{F_i} < \frac{1}{2}\left(\frac{1}{F_{i-2}}\right)$ for $i \geq 3$ and splitting off the first few terms, this time separating the odd- and even-numbered terms, gives

$$S = \sum_{i=1}^{n} \frac{1}{F_i} = \frac{1}{F_1} + \frac{1}{F_2} + \left(\frac{1}{F_3} + \frac{1}{F_5} + ...\right) + \left(\frac{1}{F_4} + \frac{1}{F_6} + ...\right)$$

$$< 1 + 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right)$$

$$= 1 + 1 + \frac{1}{2} \times 2 + \frac{1}{3} \times 2 = 3\tfrac{2}{3}.$$

(ii) To show that $S > 3.2$, we simply apply the same approaches as before, but taking more terms initially before summing our GP (or stopping at F_7 in the "simpler approach" mentioned previously). Something like

$$S > \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 = 3\tfrac{7}{30} > 3\tfrac{6}{30} = 3.2$$

does the job pretty readily. Then, to show that $S < 3\tfrac{1}{2}$, a similar argument to those you have been directed towards by the question, works well with little extra thought required:

$$S < 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) + \frac{1}{8}\left(1 + \frac{1}{2} + \frac{1}{4} + ...\right)$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 + \frac{1}{8} \times 2 = 3\tfrac{29}{60} < 3\tfrac{1}{2}.$$

Returning to the initial argument, $F_i < 2\,F_{i-1}$ or $\frac{1}{F_i} > \frac{1}{2}\left(\frac{1}{F_{i-1}}\right)$ for $i \geq 4$, we can extend this to $F_i > \frac{3}{2}\,F_{i-1}$ or $\frac{1}{F_i} < \frac{2}{3}\left(\frac{1}{F_{i-1}}\right)$ for $i \geq 5$, $F_i < \frac{5}{3}\,F_{i-1}$ or $\frac{1}{F_i} > \frac{3}{5}\left(\frac{1}{F_{i-1}}\right)$ for $i \geq 6$, etc., simply by using the defining recurrence relation for the *Fibonacci Numbers*, leading to the general results

$$F_n > \left(\frac{F_{2k}}{F_{2k-1}}\right) F_{n-1} \quad \text{or} \quad \frac{1}{F_n} < \left(\frac{F_{2k-1}}{F_{2k}}\right)\frac{1}{F_{n-1}} \quad \text{for } n \geq 2k + 1$$

and

$$F_n < \left(\frac{F_{2k+1}}{F_{2k}}\right) F_{n-1} \quad \text{or} \quad \frac{1}{F_n} > \left(\frac{F_{2k}}{F_{2k+1}}\right)\frac{1}{F_{n-1}} \quad \text{for } n \geq 2k + 2.$$

Since the terms $\frac{F_n}{F_{n-1}} \to \phi = \frac{\sqrt{5}+1}{2}$, the golden ratio, (being the positive root of the quadratic equation $x^2 = x + 1$, we can deduce the approximation $S \approx \sum_{i=1}^{n} \frac{1}{F_i} + \frac{1}{F_{n+1}}\phi^2$ since the geometric progression $1 + \frac{1}{\phi} + \frac{1}{\phi^2} + ... = \frac{1}{1 - \frac{1}{\phi}} = \frac{\phi}{\phi - 1} = \frac{\phi}{\frac{1}{\phi}} = \phi^2$. Taking $n = 9$, (i.e. just using the first 10 *Fibonacci Numbers* which you were led to write down at the start),

$$S \approx \sum_{i=1}^{9} \frac{1}{F_i} + \frac{1}{F_{10}}\phi^2 = \frac{614893}{185640} + \frac{1}{55} \times \frac{\sqrt{5}+3}{2} \approx 3.359\,89,$$

which is correct to 5 d.p. For further information on this number, try looking up the '*Reciprocal Fibonacci constant*' on Wikipedia, for instance.

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7 It is easy to saunter into this question's opening without pausing momentarily to wonder if one is going about it in the best way. Whilst many can cope with differentiating a "triple"-product with ease, many others can't. However, even for interests' sake, one might stop to consider a general approach to such matters. Differentiating $y = pqr$ (all implicitly functions of x) as, initially, $p(qr)$ and applying the product-rule twice, one obtains $y' = pq\,r' + p\,q'\,r + p'\,qr$, and this can be used here with $p = (x - a)^n$, $q = e^{bx}$ and $r = \sqrt{1+x^2}$ without the need for a lot of the mess (and subsequent mistakes) that was (were) made by so many candidates. Here, $y = (x - a)^n\, e^{bx}\sqrt{1+x^2}$ gives

$$\frac{dy}{dx} = (x-a)^n\, e^{bx}\frac{x}{\sqrt{1+x^2}} + (x-a)^n\, b\, e^{bx}\sqrt{1+x^2} + n(x-a)^{n-1}\, e^{bx}\sqrt{1+x^2}$$

Factorising out the given terms $\Rightarrow \dfrac{(x-a)^{n-1}\,e^{bx}}{\sqrt{1+x^2}}\left\{x(x-a)+b(x-a)(1+x^2)+n(1+x^2)\right\}$, and we are only required to note that the term in the brackets is, indeed, a cubic; though it may prove helpful later on to simplify it by multiplying out and collecting up terms, to get

$$q(x) = bx^3 + (n + 1 - ab)x^2 + (b - a)x + (n - ab).$$

(i) The first integral, $I_1 = \displaystyle\int \frac{(x-4)^{14}\,e^{4x}}{\sqrt{1+x^2}}\left(4x^3-1\right)\,dx$, might reasonably be expected to be a very straightforward application of the general result, and so it proves to be. With $n = 15$, and taking $a = b = 4$, so that $q(x) = 4x^3 - 1$ (which really should be checked explicitly), we find

$$I_1 = (x - 4)^{15}\, e^{4x}\sqrt{1+x^2} \ \ (+\ C).$$

(ii) This second integral, $I_2 = \displaystyle\int \frac{(x-1)^{21}\,e^{12x}}{\sqrt{1+x^2}}\left(12x^4-x^2-11\right)\,dx$, is clearly not so straightforward, since the bracketed term is now quartic. Of the many things one *might* try, however, surely the simplest is to try to factor out a linear term, the obvious candidate being $(x - 1)$.

Finding that $12x^4 - x^2 - 11 \equiv (x - 1)(12x^3 + 12x^2 + 11x + 11)$, we now try $n = 23$, $a = 1$, $b = 12$ to obtain $q(x) = 12x^3 + 12x^2 + 11x + 11$ and $I_2 = (x - 1)^{23}\, e^{12x}\sqrt{1+x^2}$ $(+\ C)$.

(iii) The final integral, $I_3 = \displaystyle\int \frac{(x-2)^{6}\,e^{4x}}{\sqrt{1+x^2}}\left(4x^4+x^3-2\right)\,dx$, is clearly intended to be even less simple than its predecessor. However, you might now suspect that "the next case up" is in there somewhere. So, if you try $n = 8$, $a = 2$, $b = 4$, which gives

$$\frac{dy_8}{dx} = \frac{(x-2)^7\,e^{4x}}{\sqrt{1+x^2}}\left\{4x^3+x^2+2x\right\} = \frac{(x-2)^6\,e^{4x}}{\sqrt{1+x^2}}\left\{4x^4-7x^3-4x\right\},$$

as well as the obvious target $n = 7$, $a = 2$, $b = 4$, which yields

$$\frac{dy_7}{dx} = \frac{(x-2)^6\,e^{4x}}{\sqrt{1+x^2}}\left\{4x^3+2x-1\right\},$$

It may now be clear that ***both*** are involved. Indeed,

$$I_3 = \int\left(\frac{dy_8}{dx} + 2\frac{dy_7}{dx}\right)dx = y_8 + 2\,y_7 = x(x-2)^7\,e^{4x}\sqrt{1+x^2} \ \ (+\ C).$$

8 For the diagram, you are simply required to show P on AB, strictly between A and B; and Q on AC on other side of A to C. The two given parameters indicate that $CQ = \mu AC$ and $BP = \lambda AB$. Substituting these into the given expression, $CQ \times BP = AB \times AC \Rightarrow \mu AC . \lambda AB = AB . AC$ $\Rightarrow \mu = \frac{1}{\lambda}$. [Notice that CQ, BP, etc., are scalar quantities, and hence the "×" **cannot** be the vector product!]

Writing the equation of line PQ in the form $\mathbf{r} = t\,\mathbf{p} + (1 - t)\,\mathbf{q}$ for some scalar parameter t and substituting the given forms for $\mathbf{p}$ and $\mathbf{q}$ gives $\mathbf{r} = t\lambda\,\mathbf{a} + t(1 - \lambda)\mathbf{b} + (1 - t)\mu\,\mathbf{a} + (1 - t)(1- \mu)\mathbf{c}$. Eliminating $\mu = \frac{1}{\lambda} \Rightarrow \mathbf{r} = \left(t\lambda + \frac{1}{\lambda} - \frac{t}{\lambda}\right)\mathbf{a} + t(1-\lambda)\mathbf{b} + (1-t)\left(\frac{\lambda - 1}{\lambda}\right)\mathbf{c}$. Comparing this to the given answer, we note that when $t = \frac{1}{1-\lambda}$ from the $\mathbf{b}$-component, $1 - t = \frac{\lambda}{\lambda - 1}$, etc., so that we do indeed get $\mathbf{r} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$, as required.

Since $\mathbf{d} - \mathbf{c} = \mathbf{b} - \mathbf{a}$, one pair of sides of opposite sides of $ABDC$ are equal and parallel, so we can conclude that $ABDC$ is a parallelogram

9 (i) If you "break the lamina up" into a rectangle and a triangle (shapes whose geometric centres should be well-known to you), with relative masses 2 and 1, and impose (mentally, at least) a coordinate system onto the diagram, then the x-coordinate of the centre of mass is given by

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{2 \times \frac{9}{2} + 1 \times 12}{3} = 7.$$

(ii) A more detailed approach, but still along similar lines, might be constructed in the following, tabular way:

Shape	Mass	Dist. c.o.m. from OZ
LH end	540ρ	7
RH end	540ρ	7
Front	$41d\rho$	$\frac{27}{2}$
Back	$40d\rho$	0
Base	$9d\rho$	$\frac{9}{2}$

Note that each mass has been calculated as area × density (ρ)

Then $x_E = \frac{2 \times (540\rho) \times 7 + 41d\rho \times \frac{27}{2} + 0 + 9dp \times \frac{9}{2}}{1080\rho + 90d\rho}$, which (after much cancelling) simplifies to

$$= \frac{2 \times 60 \times 7 + 66d}{10(12 + d)} = \frac{3(140 + 11d)}{5(12 + d)}.$$

A similar approach for the full tank gives

Object	Mass	Dist. c.o.m. from OZ
Tank	2880ρ	$\frac{27}{4}$
Water	$10800k\rho$	7

and $x_F = \frac{2880\rho \times \frac{27}{4} + 10800kp \times 7}{2880\rho + 10800k\rho} = \frac{27 + 105k}{4 + 15k}$.

10 The standard approach in collision questions is to write down the equations gained when applying the principles of *Conservation of Linear Momentum* (CLM) and *Newton's Experimental Law of Restitution* (NEL or NLR), and then what can be deduced from these.

For $P_{1,2}$: CLM $\Rightarrow m_1 u = m_1 v_1 + m_2 v_2$ and NEL $\Rightarrow eu = v_2 - v_1$.
Solving to determine the final speeds of P_1 and P_2 then yields

$$v_1 = \frac{(m_1 - em_2)}{m_1 + m_2}u \quad \text{and} \quad v_2 = \frac{m_1(1+e)}{m_1 + m_2}u\,.$$

Similarly, for $P_{4,3}$: CLM $\Rightarrow m_4 u = m_4 v_4 + m_3 v_3$ and NEL $\Rightarrow eu = v_3 - v_4$, leading to

$$v_3 = \frac{m_4(1+e)}{m_3 + m_4}u \quad \text{and} \quad v_4 = \frac{(m_4 - em_3)}{m_3 + m_4}u\,.$$

If we now write $X = OP_2$ and $Y = OP_3$ initially, and equate the times to the following collisions at O, we have

(1st collision): $\dfrac{(m_1 + m_2)X}{m_1(1+e)u} = \dfrac{(m_3 + m_4)Y}{m_4(1+e)u}$

and

(2nd collision): $\dfrac{(m_1 + m_2)X}{(m_1 - em_2)u} = \dfrac{(m_3 + m_4)Y}{(m_4 - em_3)u}$.

Cancelling u's and $(1 + e)$'s

$$\Rightarrow \frac{(m_1 + m_2)X}{m_1} = \frac{(m_3 + m_4)Y}{m_4} \quad \text{and} \quad \frac{(m_1 + m_2)X}{(m_1 - em_2)} = \frac{(m_3 + m_4)Y}{(m_4 - em_3)}\,. \qquad \textbf{(*)}$$

Dividing these two (or equating for X / Y) $\Rightarrow \dfrac{m_1 - em_2}{m_1} = \dfrac{m_4 - em_3}{m_4}$, which simplifies to

$\dfrac{m_2}{m_1} = \dfrac{m_3}{m_4}$. Finally substituting back into one of the equations **(*)** then gives

$$X\left(1 + \frac{m_2}{m_1}\right) = Y\left(1 + \frac{m_3}{m_4}\right) \;\Rightarrow\; X = Y\,.$$

Rather surprisingly, however, the momentum equations turn out to be totally unnecessary here. Consider …

Collision $P_{1,2}$: NEL $\Rightarrow eu = v_2 - v_1$

Collision $P_{4,3}$: NEL $\Rightarrow eu = v_3 - v_4$ so that $v_2 - v_1 = v_3 - v_4$ **(*)**.

Next, the two equated sets of times are $\dfrac{X}{v_2} = \dfrac{Y}{v_3}$ and $\dfrac{X}{v_1} = \dfrac{Y}{v_4} \Rightarrow X v_3 = Y v_2$ and $X v_4 = Y v_1$.

Subtracting: $X(v_3 - v_4) = Y(v_2 - v_1) \Rightarrow X = Y$ from **(*)**.

11 N2L $\Rightarrow F_T - (n+1)R = (n+1)Ma$, where F_T is the tractive, or driving, force of the engine.

Using $P = F_T \, . \, v$ then gives $a = \dfrac{\frac{P}{v} - (n+1)R}{M(n+1)}$ or $\dfrac{P-(n+1)Rv}{M(n+1)v}$. Note here that, for $a > 0$ we require $P > (n+1)Rv$.

Writing $a = \dfrac{\mathrm{d}v}{\mathrm{d}t}$ gives $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{P-(n+1)Rv}{M(n+1)v}$ which is a "variables separable" first-order differential equation: $\dfrac{M(n+1)v}{P-(n+1)Rv}\,\mathrm{d}v = \mathrm{d}t \Rightarrow \displaystyle\int_0^V \frac{M(n+1)v}{P-(n+1)Rv}\,\mathrm{d}v = \int_0^T 1.\mathrm{d}t \quad (= T).$

Some care is needed to integrate the LHS here, and the simplest approach is to use a substitution such as $s = P-(n+1)Rv$, $\mathrm{d}s = -R(n+1)\,\mathrm{d}v$ to get

$$T = \frac{M}{R}\int \frac{P-s}{s}\times\frac{\mathrm{d}s}{-R(n+1)} = \frac{-M}{(n+1)R^2}\int\left(\frac{P}{s}-1\right)\mathrm{d}s = \frac{-M}{(n+1)R^2}\left[P\ln(s)-s\right]$$

$$= \frac{-M}{(n+1)R^2}\left[P\ln\big(P-(n+1)Rv\big)-\big(P-(n+1)Rv\big)\right]_0^V$$

$$= \frac{-MP}{(n+1)R^2}\left\{\ln\big(P-(n+1)Rv\big)-P+(n+1)Rv-P\ln P+P-0\right\}$$

$$= \frac{-MP}{(n+1)R^2}\ln\left(\frac{P-(n+1)Rv}{P}\right)-\frac{MV}{R}$$

More careful algebra is still required to manipulate this into a form in which the given approximation can be used:

$$T = \frac{-MP}{(n+1)R^2}\ln\left(1-\frac{(n+1)Rv}{P}\right)-\frac{MV}{R}$$

$$\approx \frac{-MP}{(n+1)R^2}\left(-\frac{(n+1)Rv}{P}-\frac{1}{2}\left(\frac{(n+1)Rv}{P}\right)^2 .....\right)-\frac{MV}{R}$$

$$= \frac{MV}{R}+\frac{(n+1)MV^2}{2P}.....-\frac{MV}{R}$$

so that $PT \approx \frac{1}{2}(n+1)MV^2$, and this is just the statement of the *Work-Energy Principle*, namely "Work Done = Change in (Kinetic) Energy", in the case when $R = 0$.

When $R \neq 0$, WD against R = WD by engine – Gain in KE $\Rightarrow$ $(n+1)RX = PT - \frac{1}{2}(n+1)MV^2$.

[Unfortunately, a last-minute change to the wording of the question led to the omission of one of the $(n + 1)$s.]

12 (i) This whole question is something of a "one-trick" game, I'm afraid, and relies heavily on being able to spot that X is just half of a normal distribution. The *Standard Normal Distribution* N(0, 1) is given by $P(X \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$. Once the connection has been spotted, the accompanying pure maths work is fairly simple, including the sketch of the graph. This is particularly important since the function e^{kx^2} cannot be integrated analytically.

(ii) Substituting $t = 2x$, $dt = 2\, dx$ and equating to $\frac{1}{2}$ (being just the positive half of a normal), gives

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}t^2}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^\infty 2e^{-2x^2}\, dx = \frac{1}{2} \Rightarrow \int_0^\infty e^{-2x^2} dx = \frac{\sqrt{2\pi}}{4}.$$

Since total probability = 1, we have $\frac{1}{k} = \frac{\sqrt{2\pi}}{4}$ and $k = \frac{4}{\sqrt{2\pi}}$.

(iii) Thereafter, $E(X) = k \int_0^\infty x e^{-2x^2} dx = k\left[-\frac{1}{4}e^{-2x^2}\right]_0^\infty = \frac{1}{4}k = \frac{1}{\sqrt{2\pi}}$.

Also, $E(X^2) = k\int_0^\infty x \times x\, e^{-2x^2} dx = k\left\{\left[-\frac{1}{4}x e^{-2x^2}\right]_0^\infty + \int_0^\infty \frac{1}{4} e^{-2x^2} dx\right\}$ using integration by parts

$$= k\left\{0 + \frac{1}{4} \times \frac{\sqrt{2\pi}}{4}\right\} = \frac{1}{4}.$$

Then $\text{Var}(X) = E(X^2) - E^2(X) = \frac{1}{4} - \frac{1}{2\pi}$ or $\frac{\pi - 2}{4\pi}$.

(iv) For the median, we want to find the value m of x for which $\frac{1}{2} = \frac{4}{\sqrt{2\pi}} \int_0^m e^{-2x^2} dx$, and this requires to undo some of the above work in order to be able to use N(0, 1) and the statistics tables provided in the formula book.

$$\frac{1}{2} = \frac{2}{\sqrt{2\pi}} \int_0^m 2e^{-2x^2} dx = 2 \times \frac{1}{\sqrt{2\pi}} \int_0^{2m} e^{-\frac{1}{2}t^2} dt = 2\{\Phi(2m) - \tfrac{1}{2}\} \text{ or } \Phi(\tfrac{1}{2}m) = \frac{3}{4}$$

Use of the $N(0, 1)$ tables then gives $2m = 0.6745$ (0.675-ish) and $m = 0.337$ or 0.338.

13 For A: p(launch fails) = p(>1 fail) = $1 - p_0 - p_1 = 1 - q^4 - 4q^3p$

so that E(repair) = $\Sigma x\,\mathrm{p}(x) = 0.q^4 + K.4q^3p + 4K(1 - q^4 - 4q^3p)$

$$= 4K\left[q^3p + (1-q)(1+q+q^2+q^3) - 4q^3p\right]$$

$$= 4Kp\left[1+q+q^2-2q^3\right]$$

For B: p(launch fails) = p(>2 fail) = $1 - p_0 - p_1 - p_2 = 1 - q^6 - 6q^5p - 15q^4p^2$

so that E(repair) = $\Sigma x\,\mathrm{p}(x)$

$$= 0.q^6 + K.6q^5p + 2K.15q^4p^2 + 6K(1 - q^6 - 6q^5p - 15q^4p^2)$$

$$= 6K\left[q^5p + 5q^4p^2 + (1-q)(1+q+q^2+q^3+q^4+q^5) - 6q^5p - 15q^4p^2\right]$$

Extracting the p and obtaining the remaining in terms of q only,

$$= 6Kp\left[q^5 + 5q^4(1-q) + 1 + q + q^2 + q^3 + q^4 + q^5 - 6q^5 - 15q^4(1-q)\right]$$

$$= 6Kp\left[1+q+q^2+q^3-9q^4+6q^5\right]$$

Setting Rep(A) = $\frac{2}{3}$ Rep(B) $\Rightarrow$ $12Kp\left[1+q+q^2-2q^3\right] = 2Kp\left[1+q+q^2+q^3-9q^4+6q^5\right]$

Clearly, $p = 0$ is one solution and the rest simplifies to

$$0 = 3q^3(1 - 3q + 2q^2) = 3q^3(1-q)(1-2q).$$

We thus have $p = 1, 0, \frac{1}{2}$, with the 0 and 1 being rather trivial solutions.